Exercise 86

The function $C(t) = K(e^{-at} + e^{-bt})$, where a, b, and K are positive constants and $b > a$, is used to model the concentration at time t of a drug injected into the bloodstream.

(a) Show that $\lim_{t\to\infty} C(t) = 0$.

(b) Find $C'(t)$, the rate of change of drug concentration in the blood.

(c) When is this rate equal to 0?

Solution

Evaluate the limit of $C(t)$ as $t \to \infty$, noting that a, b, and K are all positive.

$$\begin{aligned}\lim_{t\to\infty} C(t) &= \lim_{t\to\infty} K(e^{-at} + e^{-bt}) \\ &= K \lim_{t\to\infty} (e^{-at} + e^{-bt}) \\ &= K \lim_{t\to\infty} \left(\frac{1}{e^{at}} + \frac{1}{e^{bt}}\right) \\ &= K\left(\frac{1}{e^{a(\infty)}} + \frac{1}{e^{b(\infty)}}\right) \\ &= K\left(\frac{1}{e^{\infty}} + \frac{1}{e^{\infty}}\right) \\ &= K(0+0) \\ &= 0\end{aligned}$$

Differentiate the concentration function.

$$\begin{aligned}C'(t) = \frac{d}{dt}[K(e^{-at} + e^{-bt})] = K\frac{d}{dt}(e^{-at} + e^{-bt}) &= K\left[\frac{d}{dt}(e^{-at}) + \frac{d}{dt}(e^{-bt})\right] \\ &= K\left[(e^{-at}) \cdot \frac{d}{dt}(-at) + (e^{-bt}) \cdot \frac{d}{dt}(-bt)\right] \\ &= K[(e^{-at}) \cdot (-a) + (e^{-bt}) \cdot (-b)] \\ &= K(-ae^{-at} - be^{-bt}) \\ &= -K(ae^{-at} + be^{-bt})\end{aligned}$$

The rate is never zero because $K > 0$, $ae^{-at} > 0$, and $be^{-bt} > 0$. It only approaches zero in the limit as $t \to \infty$.